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Microscopic Calculation of the Dielectric Susceptibility Tensor for Coulomb Fluids

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In a Coulomb fluid confined to a domain V, the dielectric susceptibility tensor χ_V depends on the shape of V, even in the thermodynamic $V \to \infty$ limit. This paper deals with the classical two-dimensional one-component plasma formulated in an elliptic V-domain, equilibrium statistical mechanics is used. For the dimensionless coupling constant $\Gamma =$ even positive integer, the mapping of the plasma onto a discrete one-dimensional anticommuting-field theory provides a new sum rule. This sum rule confirms the limiting value of χ_V predicted by macroscopic electrostatics and gives a finite-size correction term to χ_V.

KEY WORDS: One-component plasma; two dimensions; sum rules.

1. INTRODUCTION

Classical Coulomb systems are prototypes for studying the effect of long-range interactions in equilibrium statistical mechanics. In dimension ν, the Coulomb potential ϕ^c at a spatial position $\mathbf{r} = (r^1, r^2, ..., r^\nu)$, induced by a unit charge at the origin, is the solution of the Poisson equation

$$\Delta \phi^c(\mathbf{r}) = -\varepsilon_\nu \delta(\mathbf{r}) \tag{1}$$

where ε_ν is the surface area of the ν-dimensional unit sphere; $\varepsilon_2 = 2\pi$, $\varepsilon_3 = 4\pi$, etc. In particular, in two dimensions one has the logarithmic potential

$$\phi^c(\mathbf{r}) = -\ln(|\mathbf{r}|/r_0) \tag{2}$$

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where the length scale r_0 is set to unity, for simplicity. In what follows, we will restrict the discussion to two and three dimensions, so all presented formula containing ν will be valid only for $\nu = 2, 3$.

A general Coulomb system consists of s pointlike species $\alpha = 1, \ldots, s$ with the corresponding charges q_α, embedded in a fixed uniform neutralizing background of density n_0 and charge density ρ_0. The most studied one-component jellium or plasma (OCP) and two-component plasma (TCP) correspond to $s = 1$ $(q_1 = q)$, $\rho_0 = -q n_0 \neq 0$ and to $s = 2$ $(q_1 = -q_2)$, $\rho_0 = n_0 = 0$, respectively. The Coulomb system is confined to a domain V, which can be:

(1) infinite, $V \to \mathbf{R}^\nu$;

(2) finite, bounded by an impermeable hard wall (for the sake of simplicity, uncharged and with no image forces);

(3) semi-infinite, i.e., bounded by a wall, but infinite in at least one of the parallel directions.

The symbol $\langle \ldots \rangle_V$ will denote the canonical averaging over the domain V at the inverse temperature $\beta = 1/k_B T$, under the system neutrality condition. The microscopic total number and charge densities at $\mathbf{r}$ are given by

$$\hat{n}(\mathbf{r}) = \sum_i \delta(\mathbf{r} - \mathbf{r}_i) \tag{3a}$$

$$\hat{\rho}(\mathbf{r}) = \sum_i q_{\alpha_i} \delta(\mathbf{r} - \mathbf{r}_i) \tag{3b}$$

respectively, where the sums run over N particle indices. The canonical average number and charge densities read

$$n_V(\mathbf{r}) = \langle \hat{n}(\mathbf{r}) \rangle_V, \qquad \rho_V(\mathbf{r}) = \langle \hat{\rho}(\mathbf{r}) \rangle_V \tag{4}$$

At the two-particle level, one considers the two-body distribution

$$n_V(\mathbf{r}, \mathbf{r}') = \left\langle \sum_{j \neq k} \delta(\mathbf{r} - \mathbf{r}_j)\, \delta(\mathbf{r}' - \mathbf{r}_k) \right\rangle_V \tag{5a}$$

as well as its truncated form

$$n_V^T(\mathbf{r}, \mathbf{r}') = n_V(\mathbf{r}, \mathbf{r}') - n_V(\mathbf{r})\, n_V(\mathbf{r}') \tag{5b}$$

and the truncated charge–charge correlation function

$$S_V(\mathbf{r} \mid \mathbf{r}') = \langle \hat{\rho}(\mathbf{r})\, \hat{\rho}(\mathbf{r}') \rangle_V - \langle \hat{\rho}(\mathbf{r}) \rangle_V \langle \hat{\rho}(\mathbf{r}') \rangle_V \tag{6}$$

In the case of the OCP, S is expressible as follows

$$S_V(\mathbf{r} \mid \mathbf{r}') = q^2[n_V^T(\mathbf{r}, \mathbf{r}') + n_V(\mathbf{r})\, \delta(\mathbf{r}-\mathbf{r}')] \tag{7}$$

The long-range tail of the Coulomb force causes screening, and thus gives rise to exact constraints, sum rules, for the structure function S (see review of ref. 1).

In bulk regime, $\lim_{V \to \mathbf{R}^\nu} S_V(\mathbf{r} \mid \mathbf{r}') = S(|\mathbf{r}-\mathbf{r}'|)$ is known to obey the Stillinger–Lovett rules[2, 3] which imply the zeroth-moment (electroneutrality) condition

$$\int d\mathbf{r}\, S(\mathbf{r}) = 0 \tag{8}$$

and the second-moment condition

$$\begin{aligned} \beta \int d\mathbf{r} (r^i)^2\, S(\mathbf{r}) &= \frac{\beta}{\nu} \int d\mathbf{r}\, |\mathbf{r}|^2\, S(\mathbf{r}) \\ &= -\frac{1}{\pi(\nu-1)} \qquad i = 1,..., \nu \end{aligned} \tag{9}$$

For the OCP, the fourth moment of S is related to the isothermal compressibility,[4–6] so that knowledge of the exact equation of state in two dimensions[7] provides its explicit form.[8, 9] Very recently,[10] the sixth moment of S for the two-dimensional (2d) OCP was derived using a renormalized Mayer expansion in density.[11] The formal analogues of the fourth and sixth moments of S in the 2d OCP are the respective zeroth and second moments of the truncated total number density correlation function $\langle \hat{n}(\mathbf{r})\, \hat{n}(\mathbf{r}') \rangle - \langle \hat{n}(\mathbf{r}) \rangle \langle \hat{n}(\mathbf{r}') \rangle$ in the 2d TCP, as was derived in ref. 12 from analogies with critical systems and in ref. 13 directly by using diagrammatic methods.

For finite systems, the zeroth-moment sum rule

$$\int_V d\mathbf{r}\, S_V(\mathbf{r} \mid \mathbf{r}') = \int_V d\mathbf{r}'\, S_V(\mathbf{r} \mid \mathbf{r}') = 0 \tag{10}$$

only tells us that the total charge in the domain V is fixed. The information analogous to the second-moment formula (9) is contained in the dielectric susceptibility tensor χ_V, defined by

$$\chi_V^{ij} = \frac{\beta}{|V|} (\langle P^i P^j \rangle_V - \langle P^i \rangle_V \langle P^j \rangle_V) \tag{11a}$$

where

$$P^i = \int_V d\mathbf{r}\, r^i \hat{\rho}(\mathbf{r}) \qquad i = 1,..., \nu \tag{11b}$$

is the ith component of the total polarization in the system and $|V|$ is the volume. Within the linear-response theory, χ_V relates the average polarization to a constant applied field $\mathbf{E}$, $\langle P^i \rangle = \sum_{j=1}^{\nu} \chi_V^{ij} E^j$. With regard to (10), the tensor χ_V is expressible in two equivalent ways,

$$\begin{aligned}\chi_V^{ij} &= \frac{\beta}{|V|} \int_V d\mathbf{r}_1 \int_V d\mathbf{r}_2\, r_1^i r_2^j S_V(\mathbf{r}_1 \mid \mathbf{r}_2) \\ &= -\frac{\beta}{2\,|V|} \int_V d\mathbf{r}_1 \int_V d\mathbf{r}_2 (r_1^i - r_2^j)^2\, S_V(\mathbf{r}_1 \mid \mathbf{r}_2)\end{aligned} \tag{12}$$

As $V \to \mathbf{R}^\nu$ one would intuitively expect that, according to the bulk second-moment formula (9), the diagonal components $\chi_V^i = \chi_V^{ii}$ $(i = 1,..., \nu)$ tend to the value

$$\chi_V^i \to -\frac{\beta}{2} \int d\mathbf{r} (r^i)^2\, S(\mathbf{r}) = \frac{1}{2\pi(\nu - 1)} \tag{13}$$

However, this is not the case: due to surface effects, the χ_V limit depends on the shape of V. Its value is predicted by macroscopic electrostatics for homogeneously polarizable systems.[14, 15] In the case of elliptic ($\nu = 2$) and ellipsoidal ($\nu = 3$) V-domains, one introduces the depolarization tensor T_V

$$T_V^{ij} = -\frac{1}{2\pi(\nu - 1)} \frac{\partial^2}{\partial r^i\, \partial r^j} \int_V d\mathbf{r}'\, \phi^c(\mathbf{r} - \mathbf{r}') \tag{14}$$

where $\mathbf{r}$ is an arbitrary point in V. It is the fundamental property of the elliptic and ellipsoidal domains that the tensor T_V is independent of the point $\mathbf{r} \in V$, and depends only on the shape of V. With regard to the Poisson equation (1), its diagonal elements $T_V^i = T_V^{ii}$ are constrained by $\sum_{i=1}^{\nu} T_V^i = \varepsilon_\nu / [2\pi(\nu - 1)]$. In the limit $V \to \mathbf{R}^\nu$, electrostatics yields

$$\chi_V^i = \frac{1}{2\pi(\nu - 1)\, T_V^i} \tag{15}$$

In the special case of a 2d disk or 3d sphere, T_V is isotropic, so that $T_V^i = \varepsilon_\nu / [2\pi\nu(\nu - 1)]$. Consequently,

$$\chi_V^i \to \frac{\nu}{\varepsilon_\nu} = \frac{\nu}{2\pi(\nu - 1)} \tag{16}$$

in contradiction with the previously suggested estimate (13). The formula (16) was checked and finite-size corrections were calculated in refs. 14 and 15 for the 2d OCP formulated on the disk for an exactly solvable case of the dimensionless coupling $\Gamma = \beta q^2 = 2$.[16, 17]

This paper deals with the 2d OCP formulated in the elliptic domain, which includes a circularly symmetric disk and the limiting case, a strip. The statistics now depends on the only parameter—the coupling constant Γ (the particle density only scales appropriately the distance). At $\Gamma =$ even integer, the 2d OCP is mappable onto a discrete 1d anticommuting-field theory.[18, 19] It is shown that, in general, sum rules come from specific unitary transformations of anticommuting variables, keeping a specific "composite" form of the fermionic action. A nontrivial transformation of anticommuting variables is revealed to generate a new sum rule. For the elliptic domain, this sum rule confirms microscopically the asymptotic formula (15) and gives a finite-size correction term to χ_V^i explicitly in terms of boundary contributions.

The paper is organized as follows. Section 2 recapitulates briefly the mapping of the 2d OCP onto the 1d fermionic model. Section 3 establishes a formalism of the unitary transformations of anticommuting variables, which imply the known sum rules.[20] Complementary (to author's knowledge as-yet-unknown) sum rules, nontrivial when some asymmetry of the V-domain is present, are given as well. In the key Section 4, using a special "nearest-neighbor" transformation of anticommuting variables, one derives a new sum rule providing a proper split of χ_V^i into its asymptotic (15) and finite-size correction parts. In Appendix, by explicit calculations in the 2d OCP on the disk at $\Gamma = 2$, a test of the results is presented.

2. MAPPING ONTO 1d FERMIONS

The model under consideration is the 2d OCP of N particles confined to a domain V. For a point $\mathbf{r} \in V$, the cartesian (x, y), complex $(z, \bar{z})$ or polar (r, ϕ) coordinate representations will be suitably used. The neutralizing background of density $n_0 = N/|V|$ induces the one-particle potential $-q n_0 \phi_b(\mathbf{r})$ where

$$\phi_b(\mathbf{r}) = \int_V \mathrm{d}^2 r' \; \phi^c(|\mathbf{r} - \mathbf{r}'|) \tag{17}$$

For the elliptic V-domain of interest, in the reference frame defined by the axis of the ellipse, $x^2/a^2 + y^2/b^2 = 1$, both tensors χ_V and T_V are diagonal. The fundamental independence of the depolarization tensor T_V (14) of the point $\mathbf{r} \in V$ and the invariance of $\phi_b(\mathbf{r})$ with respect to the reflection along the x or y axis imply

$$\phi_b(\mathbf{r}) = \text{const} - \pi T_V^x x^2 - \pi T_V^y y^2 \tag{18a}$$

with $T_V^x = b/(a+b)$, $T_V^y = a/(a+b)$. The corresponding electric field is $-qn_0 \mathbf{E}_b(\mathbf{r})$ where

$$\mathbf{E}_b(\mathbf{r}) = -\nabla\phi_b(\mathbf{r}) = 2\pi T_V^x x \hat{\mathbf{x}} + 2\pi T_V^y y \hat{\mathbf{y}} \tag{18b}$$

$\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ being perpendicular unit vectors in x, y directions. In the circularly symmetric case of the disk, $a = b = R$ (radius), one has $T_V^x = T_V^y = 1/2$, so that

$$\phi_b(\mathbf{r}) = -\pi r^2/2, \qquad \mathbf{E}_b(\mathbf{r}) = \pi \mathbf{r} \tag{19}$$

The total Boltzmann factor associated with a particle configuration $\{\mathbf{r}_i\}$ is written as

$$\exp\left[\Gamma n_0 \sum_i \phi_b(\mathbf{r}_i) - \Gamma \sum_{i<j} \phi^c(|\mathbf{r}_i - \mathbf{r}_j|) \right] \tag{20}$$

For $\Gamma = 2\gamma$, γ being a positive integer, it was shown in ref. 18 that the canonical partition function of the 2d OCP, Z_V (we will omit in the notation the dependence on N), can be expressed in terms of Grassmann variables $\{\xi_i^{(\alpha)}, \psi_i^{(\alpha)}\}$ ($\alpha = 1,..., \gamma$), defined on the sites $i = 0, 1,..., N-1$ of a discrete chain and satisfying ordinary anticommuting algebra and integration rules,[21] as follows:

$$Z_V = \int \mathcal{D}\psi\, \mathcal{D}\xi \exp[\mathcal{S}_V(\xi, \psi)] \tag{21a}$$

$$\mathcal{S}_V(\xi, \psi) = \sum_{i,\, j=0}^{\gamma(N-1)} \Xi_i w_{ij} \Psi_j \tag{21b}$$

Here, $\mathcal{D}\psi\, \mathcal{D}\xi = \prod_{i=0}^{N-1} \mathrm{d}\psi_i^{(\gamma)} \cdots \mathrm{d}\psi_i^{(1)}\, \mathrm{d}\xi_i^{(\gamma)} \cdots \mathrm{d}\xi_i^{(1)}$ and the fermionic action $\mathcal{S}_V$ involves pair interactions of the "composite" variables

$$\Xi_i = \sum_{\substack{i_1,..., i_\gamma = 0 \\ (i_1 + \cdots + i_\gamma = i)}}^{N-1} \xi_{i_1}^{(1)} \cdots \xi_{i_\gamma}^{(\gamma)} \tag{22a}$$

$$\Psi_j = \sum_{\substack{j_1,..., j_\gamma = 0 \\ (j_1 + \cdots + j_\gamma = j)}}^{N-1} \psi_{j_1}^{(1)} \cdots \psi_{j_\gamma}^{(\gamma)} \tag{22b}$$

i.e., the products of γ anticommuting-field components with a given sum of site indices. The interaction strength is given by

$$w_{ij} = \int_V \mathrm{d}^2 z \, z^i \bar{z}^j w(z, \bar{z}) \tag{23}$$

where $w(\mathbf{r}) = \exp[\Gamma n_0 \phi_b(\mathbf{r})]$. Denoting by

$$\langle \cdots \rangle = \frac{1}{Z_V} \int \mathscr{D}\psi \, \mathscr{D}\xi \cdots \exp[\mathscr{S}_V(\xi, \psi)] \tag{24}$$

the averaging over the 1d fermionic system, the particle-number density (4) can be expressed as

$$n_V(\mathbf{r}) = w(z, \bar{z}) \sum_{i, j=0}^{\gamma(N-1)} \langle \Xi_i \Psi_j \rangle \, z^i \bar{z}^j \tag{25}$$

the two-body distribution (5a) and its truncation (5b) as

$$n_V(\mathbf{r}_1, \mathbf{r}_2) = w(z_1, \bar{z}_1) \, w(z_2, \bar{z}_2) \sum_{i_1, j_1, i_2, j_2=0}^{\gamma(N-1)} \langle \Xi_{i_1} \Psi_{j_1} \Xi_{i_2} \Psi_{j_2} \rangle \, z_1^{i_1} \bar{z}_1^{j_1} z_2^{i_2} \bar{z}_2^{j_2} \tag{26a}$$

$$n_V^T(\mathbf{r}_1, \mathbf{r}_2) = w(z_1, \bar{z}_1) \, w(z_2, \bar{z}_2) \sum_{i_1, j_1, i_2, j_2=0}^{\gamma(N-1)} \langle \Xi_{i_1} \Psi_{j_1} \Xi_{i_2} \Psi_{j_2} \rangle^T \, z_1^{i_1} \bar{z}_1^{j_1} z_2^{i_2} \bar{z}_2^{j_2} \tag{26b}$$

where $\langle \Xi_{i_1} \Psi_{j_1} \Xi_{i_2} \Psi_{j_2} \rangle^T = \langle \Xi_{i_1} \Psi_{j_1} \Xi_{i_2} \Psi_{j_2} \rangle - \langle \Xi_{i_1} \Psi_{j_1} \rangle \langle \Xi_{i_2} \Psi_{j_2} \rangle$. For the disk (19), since the Boltzmann weight $w(\mathbf{r})$ possesses circular symmetry, the interaction matrix w_{ij} takes the diagonal form,

$$w_{ij} = \delta_{ij} w_i, \qquad w_i = \int_V \mathrm{d}^2 r \, r^{2i} w(r) \tag{27}$$

Owing to the "diagonalization" of the action, $\mathscr{S}_V = \sum_i \Xi_i w_i \Psi_i$, only the fermionic correlators $\langle \Xi_{i_1} \Psi_{j_1} \Xi_{i_2} \Psi_{j_2} \cdots \rangle$ with $i_1 + i_2 + \cdots = j_1 + j_2 + \cdots$ survive.

3. ORDINARY SUM RULES AND THEIR COMPLEMENTS

Sum rules result from the fermionic representation of the 2d OCP by specific transformations of anticommuting variables, keeping the composite nature of the action $\mathscr{S}_V$ (21b).

Let us first rescale by a constant one of the field components, say $\{\xi^{(1)}\}$,

$$\xi_i^{(1)} \to \mu \xi_i^{(1)} \qquad i=0, 1,..., N-1 \tag{28}$$

Jacobian of the transformation equals to μ^N and the fermionic action $\mathscr{S}_V$ transforms to $\mu \mathscr{S}_V$. Consequently,

$$Z_V = \mu^{-N} \int \mathscr{D}\psi \, \mathscr{D}\xi \, \exp\left(\mu \sum_{i,\, j=0}^{\gamma(N-1)} \Xi_i w_{ij} \Psi_j \right) \tag{29a}$$

$$Z_V \langle \Xi_i \Psi_j \rangle = \mu^{-N+1} \int \mathscr{D}\psi \, \mathscr{D}\xi \, \Xi_i \Psi_j \exp\left(\mu \sum_{k,\, l=0}^{\gamma(N-1)} \Xi_k w_{kl} \Psi_l \right) \tag{29b}$$

etc. Z_V, a Grassmanian integral, is independent of μ, thus its derivative with respect to μ is zero for any value of μ. For the special case $\mu=1$, the equality $\partial \ln Z_V / \partial \mu |_{\mu=1} = 0$ implies

$$-N + \sum_{i,\, j=0}^{\gamma(N-1)} w_{ij} \langle \Xi_i \Psi_j \rangle = 0 \tag{30}$$

which, after substituting (23), regarding (25) and setting $N = n_0 \, |V|$, results in the trivial neutrality condition

$$\int_V \mathrm{d}^2 r [n_V(\mathbf{r}) - n_0] = 0 \tag{31}$$

Analogously, the equality $\partial [Z_V \langle \Xi_i \Psi_j \rangle] / \partial \mu |_{\mu=1} = 0$ yields

$$-(N-1)\langle \Xi_i \Psi_j \rangle + \sum_{k,\, l=0}^{\gamma(N-1)} w_{kl} \langle \Xi_i \Psi_j \Xi_k \Psi_l \rangle = 0 \tag{32}$$

which is readily shown to be equivalent to the neutrality relation (10).

Let us now consider another linear transformation of *all* ξ-field components,

$$\xi_i^{(\alpha)} \to \lambda^i \xi_i^{(\alpha)} \quad i=0, 1,..., N-1; \qquad \alpha = 1,..., \gamma \tag{33a}$$

or *all* ψ-field components,

$$\psi_j^{(\alpha)} \to \lambda^j \psi_j^{(\alpha)} \quad j=0, 1,..., N-1; \qquad \alpha = 1,..., \gamma \tag{33b}$$

Jacobian of both transformations equals to $\lambda^{\gamma N(N-1)/2}$ and the action transforms as $\mathscr{S}_V \to \sum_{i,j=0}^{\gamma(N-1)} \lambda^i \Xi_i w_{ij} \Psi_j$ under (33a) and as $\mathscr{S}_V \to \sum_{i,j=0}^{\gamma(N-1)} \lambda^j \Xi_i w_{ij} \Psi_j$ under (33b). Thus,

$$Z_V = \lambda^{-\gamma N(N-1)/2} \int \mathscr{D}\psi \, \mathscr{D}\xi \exp\left(\sum_{i,j=0}^{\gamma(N-1)} \lambda^i \Xi_i w_{ij} \Psi_j \right)$$
$$Z_V = \lambda^{-\gamma N(N-1)/2} \int \mathscr{D}\psi \, \mathscr{D}\xi \exp\left(\sum_{i,j=0}^{\gamma(N-1)} \lambda^j \Xi_i w_{ij} \Psi_j \right) \tag{34a}$$

$$Z_V \langle \Xi_i \Psi_j \rangle = \lambda^{-\gamma N(N-1)/2+i} \int \mathscr{D}\psi \, \mathscr{D}\xi \, \Xi_i \Psi_j \exp\left(\sum_{k,l=0}^{\gamma(N-1)} \lambda^k \Xi_k w_{kl} \Psi_l \right)$$
$$Z_V \langle \Xi_i \Psi_j \rangle = \lambda^{-\gamma N(N-1)/2+j} \int \mathscr{D}\psi \, \mathscr{D}\xi \, \Xi_i \Psi_j \exp\left(\sum_{k,l=0}^{\gamma(N-1)} \lambda^l \Xi_k w_{kl} \Psi_l \right) \tag{34b}$$

The equality $\partial \ln Z_V / \partial \lambda |_{\lambda=1} = 0$ implies

$$-\tfrac{1}{2} \gamma N(N-1) + \sum_{i,j=0}^{\gamma(N-1)} i w_{ij} \langle \Xi_i \Psi_j \rangle = 0 \tag{35a}$$

$$-\tfrac{1}{2} \gamma N(N-1) + \sum_{i,j=0}^{\gamma(N-1)} j w_{ij} \langle \Xi_i \Psi_j \rangle = 0 \tag{35b}$$

On account of (30), this is equivalent to the couple of complex-conjugate equations

$$\tfrac{1}{2} \gamma N(N-1) + N = \int d^2 z \, w(z, \bar{z}) \sum_{i,j=0}^{\gamma(N-1)} \langle \Xi_i \Psi_j \rangle \, \partial^+ (z^{i+1} \bar{z}^j) \tag{36a}$$

$$\tfrac{1}{2} \gamma N(N-1) + N = \int d^2 z \, w(z, \bar{z}) \sum_{i,j=0}^{\gamma(N-1)} \langle \Xi_i \Psi_j \rangle \, \partial^- (z^i \bar{z}^{j+1}) \tag{36b}$$

where we have introduced the derivative operators

$$\partial^+ = \frac{1}{2} \left(\frac{\partial}{\partial x} + \frac{1}{i} \frac{\partial}{\partial y} \right) \qquad \partial^- = \frac{1}{2} \left(\frac{\partial}{\partial x} - \frac{1}{i} \frac{\partial}{\partial y} \right) \tag{37}$$

$(\partial^+ \equiv \partial_z, \partial^- \equiv \partial_{\bar{z}})$. They act on complex coordinates according to

$$\partial^+ z = 1, \quad \partial^+ \bar{z} = 0; \qquad \partial^- z = 0, \quad \partial^- \bar{z} = 1$$

With $\ln w(z,\bar{z}) = \Gamma n_0 \phi_b(\mathbf{r})$, ϕ_b given by (17), it is easy to verify validity of the equalities

$$\int_V \mathrm{d}^2 z [\partial^+ \ln w(z,\bar{z})]\, z n_0 = \int_V \mathrm{d}^2 z [\partial^- \ln w(z,\bar{z})]\, \bar{z} n_0 = -\tfrac{1}{2}\gamma N^2 \tag{38}$$

Then, after some algebra, Eqs. (36) take the form

$$N\left(1-\frac{\gamma}{2}\right) = \int_V \mathrm{d}^2 z\, \partial^+ [z n(z,\bar{z})] - \int_V \mathrm{d}^2 z [\partial^+ \ln w(z,\bar{z})]\, z\, \delta n_V(z,\bar{z}) \tag{39a}$$

$$N\left(1-\frac{\gamma}{2}\right) = \int_V \mathrm{d}^2 z\, \partial^- [\bar{z} n(z,\bar{z})] - \int_V \mathrm{d}^2 z [\partial^- \ln w(z,\bar{z})]\, \bar{z}\, \delta n_V(z,\bar{z}) \tag{39b}$$

with $\delta n_V(z,\bar{z}) = n_V(z,\bar{z}) - n_0$. Let us denote by ∂V the positively oriented contour enclosing the domain V: ∂V is defined parametrically as follows $x = X(\phi)$, $y = Y(\phi)$; $\phi_0 \leqslant \phi \leqslant \phi_1$. In particular, the ellipse contour admits the parametrization $X(\phi) = a\cos\phi$, $Y = b\sin\phi$; $0 \leqslant \phi \leqslant 2\pi$. Integrals over the V-domain can be expressed in terms of the ∂V-contour integrals according to the formula

$$\int_V \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\, \mathrm{d}y = \int_{\partial V} (P\, \mathrm{d}x + Q\, \mathrm{d}y) \tag{40a}$$

where

$$\begin{aligned} \int_{\partial V} P(x,y)\, \mathrm{d}x &= \int_{\phi_0}^{\phi_1} \mathrm{d}\phi\, P[X(\phi), Y(\phi)]\, X'(\phi) \\ \int_{\partial V} Q(x,y)\, \mathrm{d}y &= \int_{\phi_0}^{\phi_1} \mathrm{d}\phi\, Q[X(\phi), Y(\phi)]\, Y'(\phi) \end{aligned} \tag{40b}$$

Thus, summing and subtracting Eqs. (39a) and (39b), one gets respectively

$$\Gamma n_0 \int_V \mathrm{d}^2 r [\mathbf{r}\cdot\mathbf{E}_b(\mathbf{r})]\, \delta n_V(\mathbf{r}) = N\left(2-\frac{\Gamma}{2}\right) - \int_{\phi_0}^{\phi_1} \mathrm{d}\phi\, n_V(X,Y)(XY' - X'Y) \tag{41a}$$

$$\Gamma n_0 \int_V \mathrm{d}^2 r [\mathbf{r}\times\mathbf{E}_b(\mathbf{r})]_z\, \delta n_V(\mathbf{r}) = \int_{\phi_0}^{\phi_1} \mathrm{d}\phi\, n_V(X,Y)(XX' + YY') \tag{41b}$$

where $[\mathbf{r}\times\mathbf{E}_b]_z = xE_b^y - yE_b^x$. Equation (41a) was obtained for the disk (see Eq. (4.16) in ref. 20) and represents a generalization of the contact theorem for the plane hard wall.[22, 23] The last theorem results from (41a) by moving the origin to the boundary and in the radius $R \to \infty$ limit. The new complementary relation (41b) is informative for a generally deformed domain V.

The equality $\partial[Z_V \langle \Xi_i \Psi_j \rangle]/\partial\lambda|_{\lambda=1} = 0$ results in

$$\left[-\tfrac{1}{2}\gamma N(N-1) + i\right]\langle \Xi_i \Psi_j \rangle + \sum_{k,\, l=0}^{\gamma(N-1)} k w_{kl} \langle \Xi_i \Psi_j \Xi_k \Psi_l \rangle = 0 \tag{42a}$$

$$\left[-\tfrac{1}{2}\gamma N(N-1) + j\right]\langle \Xi_i \Psi_j \rangle + \sum_{k,\, l=0}^{\gamma(N-1)} l w_{kl} \langle \Xi_i \Psi_j \Xi_k \Psi_l \rangle = 0 \tag{42b}$$

These relations can be rewritten with the aid of Eqs. (30), (32) and (35) as follows

$$(i+1)\langle \Xi_i \Psi_j \rangle = -\sum_{k,\, l=0}^{\gamma(N-1)} (k+1)\, w_{kl} \langle \Xi_i \Psi_j \Xi_k \Psi_l \rangle^T \tag{43a}$$

$$(j+1)\langle \Xi_i \Psi_j \rangle = -\sum_{k,\, l=0}^{\gamma(N-1)} (l+1)\, w_{kl} \langle \Xi_i \Psi_j \Xi_k \Psi_l \rangle^T \tag{43b}$$

It is a simple task to pass from (43) to

$$w(\mathbf{r}_1)\, \partial_1^+ \left[\frac{n_V(\mathbf{r}_1)\, z_1}{w(\mathbf{r}_1)}\right] = -\int_V \mathrm{d}^2 r_2\, w(\mathbf{r}_2)\, \partial_2^+ \left[\frac{n_V^T(\mathbf{r}_1, \mathbf{r}_2)\, z_2}{w(\mathbf{r}_2)}\right] \tag{44a}$$

$$w(\mathbf{r}_1)\, \partial_1^- \left[\frac{n_V(\mathbf{r}_1)\, \bar{z}_1}{w(\mathbf{r}_1)}\right] = -\int_V \mathrm{d}^2 r_2\, w(\mathbf{r}_2)\, \partial_2^- \left[\frac{n_V^T(\mathbf{r}_1, \mathbf{r}_2)\, \bar{z}_2}{w(\mathbf{r}_2)}\right] \tag{44b}$$

with the obvious generalization of operators (37):

$$\partial_i^+ z_j = \delta_{ij}, \quad \partial_i^+ \bar{z}_j = 0; \qquad \partial_i^- z_j = 0, \quad \partial_i^- \bar{z}_j = \delta_{ij}$$

Summing and subtracting (44a) and (44b) one finally arrives at

$$\begin{aligned} &\beta n_0 \int_V \mathrm{d}^2 r_2 [\mathbf{r}_2 \cdot \mathbf{E}_b(\mathbf{r}_2)]\, S_V(\mathbf{r}_1 \mid \mathbf{r}_2) \\ &\quad = -2 n_V(\mathbf{r}_1) - \mathbf{r}_1 \cdot \nabla_1 n_V(\mathbf{r}_1) - \int_{\phi_0}^{\phi_1} \mathrm{d}\phi\, n_V^T[\mathbf{r}_1; (X, Y)](XY' - X'Y) \end{aligned} \tag{45a}$$

$$\begin{aligned} &\beta n_0 \int_V \mathrm{d}^2 r_2 [\mathbf{r}_2 \times \mathbf{E}_b(\mathbf{r}_2)]_z\, S_V(\mathbf{r}_1 \mid \mathbf{r}_2) \\ &\quad = -(\mathbf{r}_1 \times \nabla_1)_z\, n_V(\mathbf{r}_1) + \int_{\phi_0}^{\phi_1} \mathrm{d}\phi\, n_V^T[\mathbf{r}_1; (X, Y)](XX' + YY') \end{aligned} \tag{45b}$$

Relation (45a) with $\mathbf{r}_1=\mathbf{0}$ was derived for the disk in ref. 20 [see Eq. (4.25)]. In the $R\to\infty$ limit of the disk, it is related to the dipole sum rule for the plane hard wall.[24] The complementary Eq. (45b) is new.

4. NEW SUM RULE

Let us pose the following question: provided that the anticommuting fields under consideration $\{\xi_i^{(\alpha)}\}_{\alpha=1}^{\gamma}$ are mapped onto $\{\xi_i^{(\alpha)}(t)\}_{\alpha=1}^{\gamma}$ by the nearest-neighbor transformation

$$\frac{\partial\xi_i^{(\alpha)}(t)}{\partial t}=a_i\xi_{i+1}^{(\alpha)}(t)+b_i\xi_{i-1}^{(\alpha)}(t),\qquad \xi_i^{(\alpha)}(t=0)=\xi_i^{(\alpha)}\qquad (i=0,1,\dots,N-1) \tag{46}$$

with $a_{N-1}=b_0=0$ and t being a free parameter, does there exist a choice of the coefficients $\{a_i,b_i\}$ for which also the composite variables $\{\Xi_i\}$ (22a) transform themselves according to the nearest-neighbor scheme

$$\frac{\partial\Xi_i(t)}{\partial t}=\tilde{a}_i\Xi_{i+1}(t)+\tilde{b}_i\Xi_{i-1}(t),\qquad \Xi_i(t=0)=\Xi_i\qquad [i=0,1,\dots,\gamma(N-1)] \tag{47}$$

with $\tilde{a}_{\gamma(N-1)}=\tilde{b}_0=0$? The answer is affirmative[19]: it can be proven by a direct computation that if one chooses in (46)

$$a_i=A(i+1)\qquad b_i=B(N-i) \tag{46'}$$

the consequent $\{\Xi_i(t)\}$ fulfil the differential Eq. (47) with

$$\tilde{a}_i=A(i+1)\qquad \tilde{b}_i=B[\gamma(N-1)+1-i] \tag{47'}$$

Writting formally the solution of (46) as

$$\xi_i^{(\alpha)}(t)=\sum_{j=0}^{N-1}c_{ij}(t)\,\xi_j^{(\alpha)} \tag{48}$$

there holds

$$\frac{\partial c_{ij}(t)}{\partial t}=a_ic_{i+1,j}(t)+b_ic_{i-1,j}(t),\qquad c_{ij}(0)=\delta_{ij} \tag{49}$$

Jacobian of the mapping equals to $\det c_{ij}(t)|_{i,\,j=0}^{N-1} \equiv |\mathbf{c}|$ for each of the $\xi^{(\alpha)}$-components. Its derivative with respect to t is given by

$$\frac{\partial}{\partial t}\,|\mathbf{c}| = \sum_{i,\,j=0}^{N-1} \frac{\partial c_{ij}}{\partial t}\, C_{ij} \tag{50}$$

where $C_{ij}(t)$ is the cofactor of element $c_{ij}(t)$. In combination with Eq. (49), the orthogonality condition

$$\sum_{j=0}^{N-1} c_{kj}\, C_{ij} = \delta_{ik}\, |\mathbf{c}| \tag{51}$$

thus leads to $\partial\,|\mathbf{c}|/\partial t = 0$, i.e., $|\mathbf{c}| = \text{const} = 1$ for each $\xi^{(\alpha)}$-component. We conclude that Jacobian $= 1$. For our purpose it is sufficient to consider the transformation (46), (47) with $A = 1$ and $B = 0$; the explicit solution reads

$$\xi_i^{(\alpha)}(t) = \sum_{j=i}^{N-1} \binom{j}{i}\, t^{j-i} \xi_j^{(\alpha)}, \qquad \Xi_i(t) = \sum_{j=i}^{\gamma(N-1)} \binom{j}{i}\, t^{j-i}\, \Xi_j \tag{52}$$

The insertion of the transformation (52) into the partition function,

$$\begin{aligned} Z_V &= \int \mathscr{D}\psi\; \mathscr{D}\xi(t)\, \exp\left\{ \sum_{i,\,j=0}^{\gamma(N-1)} \Xi_i(t)\, w_{ij}\, \Psi_j \right\} \\ &= \int \mathscr{D}\psi\; \mathscr{D}\xi\, \exp\left\{ \sum_{i,\,j=0}^{\gamma(N-1)} [\Xi_i + t(i+1)\, \Xi_{i+1} + O(t^2)]\, w_{ij}\, \Psi_j \right\} \end{aligned} \tag{53}$$

with $\Xi_{\gamma(N-1)+1} \equiv 0$ automatically assumed, and the consequent application of the condition $\partial \ln Z_V/\partial t|_{t=0} = 0$ lead to

$$\sum_{i,\,j=0}^{\gamma(N-1)} (i+1)\, w_{ij} \langle \Xi_{i+1} \Psi_j \rangle = 0 \tag{54a}$$

$$\sum_{i,\,j=0}^{\gamma(N-1)} (j+1)\, w_{ij} \langle \Xi_i \Psi_{j+1} \rangle = 0 \tag{54b}$$

where the second formula originates from the t-transformation of $\{\psi^{(\alpha)}\}$ anticommuting fields. The consequent couple of complex-conjugate equations

$$\int_V \mathrm{d}^2 r [\partial^+ \ln w(\mathbf{r})]\, n_V(\mathbf{r}) = \int_V \mathrm{d}^2 r\, \partial^+ n_V(\mathbf{r}) \tag{55a}$$

$$\int_V \mathrm{d}^2 r [\partial^- \ln w(\mathbf{r})]\, n_V(\mathbf{r}) = \int_V \mathrm{d}^2 r\, \partial^- n_V(\mathbf{r}) \tag{55b}$$

is expressible by using the previously developed formalism as follows

$$\Gamma n_0 \int_V \mathrm{d}^2 r\, E_b^x(\mathbf{r})\, n_V(\mathbf{r}) = -\int_{\phi_0}^{\phi_1} \mathrm{d}\phi\, n_V(X, Y)\, Y' \tag{56a}$$

$$\Gamma n_0 \int_V \mathrm{d}^2 r\, E_b^y(\mathbf{r})\, n_V(\mathbf{r}) = \int_{\phi_0}^{\phi_1} \mathrm{d}\phi\, n_V(X, Y)\, X' \tag{56b}$$

The t-independence of

$$Z_V \langle \Xi_i(t)\, \Psi_j \rangle_t = \int \mathscr{D}\psi\, \mathscr{D}\xi [\Xi_i + t(i+1)\, \Xi_{i+1} + O(t^2)]\, \Psi_j \\ \times \exp\left\{ \sum_{k,\, l=0}^{\gamma(N-1)} [\Xi_k + t(k+1)\, \Xi_{k+1} + O(t^2)]\, w_{kl} \Psi_l \right\} \tag{57}$$

and similarly of $Z_V \langle \Xi_i \Psi_j(t) \rangle_t$ manifests itself at the lowest t^1 level as

$$(i+1)\langle \Xi_{i+1} \Psi_j \rangle + \sum_{k,\, l=0}^{\gamma(N-1)} (k+1)\, w_{kl} \langle \Xi_i \Psi_j \Xi_{k+1} \Psi_l \rangle = 0 \tag{58a}$$

$$(j+1)\langle \Xi_i \Psi_{j+1} \rangle + \sum_{k,\, l=0}^{\gamma(N-1)} (l+1)\, w_{kl} \langle \Xi_i \Psi_j \Xi_k \Psi_{l+1} \rangle = 0 \tag{58b}$$

Due to (54), the four-correlators $\langle \Xi_i \Psi_j \Xi_{k+1} \Psi_l \rangle$ in (58a) and $\langle \Xi_i \Psi_j \Xi_k \Psi_{l+1} \rangle$ in (58b) can be substituted by the truncated ones $\langle \Xi_i \Psi_j \Xi_{k+1} \Psi_l \rangle^T$ and $\langle \Xi_i \Psi_j \Xi_k \Psi_{l+1} \rangle^T$, respectively. Equations (58a) and (58b) are thus expressible as

$$w(\mathbf{r}_1)\, \partial_1^+ \left[\frac{n_V(\mathbf{r}_1)}{w(\mathbf{r}_1)} \right] = -\int_V \mathrm{d}^2 r_2\, w(\mathbf{r}_2)\, \partial_2^+ \left[\frac{n_V^T(\mathbf{r}_1, \mathbf{r}_2)}{w(\mathbf{r}_2)} \right] \tag{59a}$$

$$w(\mathbf{r}_1)\, \partial_1^- \left[\frac{n_V(\mathbf{r}_1)}{w(\mathbf{r}_1)} \right] = -\int_V \mathrm{d}^2 r_2\, w(\mathbf{r}_2)\, \partial_2^- \left[\frac{n_V^T(\mathbf{r}_1, \mathbf{r}_2)}{w(\mathbf{r}_2)} \right] \tag{59b}$$

These relations can be further simplified to the form

$$\int_V \mathrm{d}^2 r_2 [\partial_2^+ \ln w(\mathbf{r}_2)]\, S_V(\mathbf{r}_1 \mid \mathbf{r}_2)/q^2 = \partial_1^+ n_V(\mathbf{r}_1) + \int_V \mathrm{d}^2 r_2\, \partial_2^+ n_V^T(\mathbf{r}_1, \mathbf{r}_2) \tag{60a}$$

$$\int_V \mathrm{d}^2 r_2 [\partial_2^- \ln w(\mathbf{r}_2)]\, S_V(\mathbf{r}_1 \mid \mathbf{r}_2)/q^2 = \partial_1^- n_V(\mathbf{r}_1) + \int_V \mathrm{d}^2 r_2\, \partial_2^- n_V^T(\mathbf{r}_1, \mathbf{r}_2) \tag{60b}$$

For the elliptic V-domain of interest it holds $\ln w(\mathbf{r}) = \Gamma n_0(\text{const} - \pi T_V^x x^2 - \pi T_V^y y^2)$. Summing and subtracting Eqs. (60a) and (60b) one then finds

$$-2\pi\beta n_0 T_V^x \int_V \mathrm{d}^2 r_2\, x_2 S_V(\mathbf{r}_1 \mid \mathbf{r}_2) = \frac{\partial}{\partial x_1} n_V(\mathbf{r}_1) + \int_V \mathrm{d}^2 r_2 \frac{\partial}{\partial x_2} n_V^T(\mathbf{r}_1, \mathbf{r}_2) \tag{61a}$$

$$-2\pi\beta n_0 T_V^y \int_V \mathrm{d}^2 r_2\, y_2 S_V(\mathbf{r}_1 \mid \mathbf{r}_2) = \frac{\partial}{\partial y_1} n_V(\mathbf{r}_1) + \int_V \mathrm{d}^2 r_2 \frac{\partial}{\partial y_2} n_V^T(\mathbf{r}_1, \mathbf{r}_2) \tag{61b}$$

respectively.

To get the diagonal elements of the dielectric susceptibility tensor χ_V (12), one applies $\int_V \mathrm{d}^2 r_1\, x_1$ to (61a) and $\int_V \mathrm{d}^2 r_1\, y_1$ to (61b), with the result

$$\chi_V^x = \frac{1}{2\pi T_V^x} - \frac{a+b}{ab} \frac{1}{2\pi^2 n_0 q^2} \int_0^{2\pi} \mathrm{d}\phi \int_V \mathrm{d}^2 r_1\, x_1 S_V[\mathbf{r}_1 \mid (X, Y)] \cos\phi \tag{62a}$$

$$\chi_V^y = \frac{1}{2\pi T_V^y} - \frac{a+b}{ab} \frac{1}{2\pi^2 n_0 q^2} \int_0^{2\pi} \mathrm{d}\phi \int_V \mathrm{d}^2 r_1\, y_1 S_V[\mathbf{r}_1 \mid (X, Y)] \sin\phi \tag{62b}$$

Here, integration per partes was combined with the sum rule (31) to obtain

$$\int_V \mathrm{d}^2 r_1\, x_1 \frac{\partial}{\partial x_1} n_V(\mathbf{r}_1) = \int_V \mathrm{d}^2 r_1 \frac{\partial}{\partial x_1} [x_1 n_V(\mathbf{r}_1)] - n_0 |V|$$

$|V| = \pi ab$, and analogously for the y-component. Equations (62) become simpler in the symmetric case of the disk of radius R, $\chi_V^x = \chi_V^y = \bar{\chi}_V$,

$$\begin{aligned}
\bar{\chi}_V &= \frac{1}{\pi} - \frac{1}{R} \frac{1}{\pi n_0 q^2} \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \int_V \mathrm{d}^2 r_1\, S_V[(r_1, \phi_1) \mid (R, \phi)]\, r_1 \cos(\phi_1 - \phi) \\
&= \frac{1}{\pi} - \frac{1}{R} \frac{1}{\pi n_0 q^2} \int_V \mathrm{d}^2 r\, x S_V[\mathbf{r} \mid (R, 0)] \\
&= \frac{1}{\pi} - \frac{1}{R} \frac{1}{\pi n_0 q^2} \int_V \mathrm{d}^2 r\, y S_V[\mathbf{r} \mid (0, R)]
\end{aligned} \tag{63}$$

where the dependence of $S_V[(r_1, \phi_1) \mid (R, \phi)]$ on the angle difference $|\phi_1 - \phi|$ was taken into account. Equation (63) can be formally rewritten as

$$\bar{\chi}_V = \frac{1}{\pi} - \frac{1}{R} \frac{1}{\pi n_0 q^2} \langle P^x \hat{\rho}[(R, 0)] \rangle_V^T$$
$$= \frac{1}{\pi} - \frac{1}{R} \frac{1}{\pi n_0 q^2} \langle P^y \hat{\rho}[(0, R)] \rangle_V^T \tag{64}$$

The final results (62)–(64) mean an explicit split of χ_V^i into its asymptotic $1/(2\pi T_V^i)$ part [see the prediction (15) of macroscopic electrostatics] and the finite-size correction term. To show this fact for the disk, with regard to the sum rule (10) one can write

$$\int_V \mathrm{d}^2 r\, x S_V[\mathbf{r} \mid (R, 0)] = -\int_V \mathrm{d}^2 r (R - x)\, S_V[\mathbf{r} \mid (R, 0)] \tag{65}$$

Moving the origin to the boundary, $x' = R - x$ and $y' = y$, the integrals on the rhs of (63) reflect the dipole moment seen from the boundary, which is known to converge to a finite value in the thermodynamic limit. We therefore conclude that the correction term $\sim 1/R$. Owing to a slow power-law decay of correlations along a plane wall,[25, 26] one has to be cautious when identifying the integrals of type (65) with their asymptotic hard-wall counterparts. Possible vagaries and a check of Eq. (63) are documented in Appendix via the exactly solvable 2d OCP at coupling $\Gamma = 2$.

In conclusion, although the above results (62)–(64) were derived strictly for the coupling constant $\Gamma = 2 *$ positive integer, it is reasonable to suppose their validity for an arbitrary Γ in the whole fluid regime. The extension of the treatment to the case of a charged wall and in the presence of image forces is straightforward. A potential generalization of the results to higher dimensions and to the TCP requires, due to a lack of the fermionic formalism, to search for a new method which, on the one hand, reproduces our findings for the 2d OCP and, on the other hand, admits a more general applicability like the linear-response theory.

APPENDIX

When $\Gamma = 2$ $(\gamma = 1)$, the 2d OCP on the disk of radius R is exactly solvable.[16, 17, 25] The fermionic correlators are given by

$$\langle \Xi_i \Psi_j \rangle = \frac{1}{w_i} \delta_{ij}, \qquad \langle \Xi_i \Psi_j \Xi_k \Psi_l \rangle = \frac{1}{w_i w_k} (\delta_{ij} \delta_{kl} - \delta_{il} \delta_{jk}) \tag{A1}$$

etc., where $\{w_i\}$ (27) are diagonal interaction strengths,

$$w_i = \int_V \mathrm{d}^2 r\, r^{2i} w(r) = \pi \int_0^N \mathrm{d}t\, t^i \exp(-t) \tag{A2}$$

written in the units of $\pi n_0 = 1$. The dielectric susceptibility tensor is expressible as

$$\bar{\chi}_V = \frac{1}{\pi R^2} \,\mathrm{Re} \left\{ \int_V \mathrm{d}^2 r\, r^2 n(r) + \int_V \mathrm{d}^2 r_1 \int_V \mathrm{d}^2 r_2\, \bar{z}_1 z_2 n_V^T(\mathbf{r}_1, \mathbf{r}_2) \right\} \tag{A3}$$

It is straightforward to verify validity of the relations

$$\int_V \mathrm{d}^2 r\, r^2 n(r) = \sum_{i=0}^{N-1} \frac{w_{i+1}}{w_i} \tag{A4a}$$

$$\int_V \mathrm{d}^2 r_1 \int_V \mathrm{d}^2 r_2\, \bar{z}_1 z_2 n_V^T(\mathbf{r}_1, \mathbf{r}_2) = -\sum_{i=1}^{N-1} \frac{w_i}{w_{i-1}} \tag{A4b}$$

so that $\bar{\chi}_V = w_N/(\pi N w_{N-1})$. By integration per partes one derives $w_i = i w_{i-1} - \pi N^i \exp(-N)$. Consequently,

$$\bar{\chi}_V = \frac{1}{\pi} - \frac{N^{N-1} \exp(-N)}{w_{N-1}} \tag{A5}$$

On the other hand,

$$\frac{1}{q^2} \int_V \mathrm{d}^2 r\, x S_V[\mathbf{r} \mid (R, 0)] = R n(R) + \int_V \mathrm{d}^2 r\, x n_V^T[\mathbf{r}; (R, 0)] \tag{A6}$$

Since

$$R n(R) = w(R) \sum_{i=0}^{N-1} \frac{R^{2i+1}}{w_i} \tag{A7a}$$

$$\int_V \mathrm{d}^2 r\, x n_V^T[\mathbf{r}; (R, 0)] = -w(R) \sum_{i=1}^{N-1} \frac{R^{2i-1}}{w_{i-1}} \tag{A7b}$$

one arrives at

$$\frac{1}{q^2} \int_V \mathrm{d}^2 r\, x S_V[\mathbf{r} \mid (R, 0)] = \frac{N^N \exp(-N)}{R w_{N-1}} \tag{A8}$$

Inserting (A8) into (63), the exact result (A5) is recovered (in the units of $\pi n_0 = 1$).

As $N \to \infty$, the asymptotic form of w_{N-1} (A2) can be calculated by the saddle-point method in the gaussian approximation:

$$w_{N-1} \sim \frac{\pi^{3/2}}{\sqrt{2}} R N^{N-1} \exp(-N) \left[1 + O\left(\frac{1}{R}\right)\right] \tag{A9}$$

Consequently, the quantity (A8), transcribed according to (65), acquires the finite value

$$\lim_{N\to\infty} -\frac{1}{q^2} \int_V \mathrm{d}^2 r (R-x)\, S_V[\mathbf{r} \mid (R,0)] = \frac{\sqrt{2}}{\pi^{3/2}} \tag{A10}$$

as was expected.

One may be tempted to identify formula (A10) with its obvious plane hard-wall counterpart. Using the explicit result[25] for the hard wall localized at $x=0$ (plasma appears in the half-space $x \geqslant 0$),

$$n(x) = n_0 \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{\exp[-(t - x\sqrt{2})^2]}{1+\phi(t)}\, \mathrm{d}t \tag{A11a}$$

$$\begin{aligned} n^T(x_1, x_2; |y_1 - y_2|) &= -n_0^2 \exp[-(x_1-x_2)^2] \\ &\times \left| \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{\exp\{-[t-(x_1+x_2)/\sqrt{2}]^2 - \mathrm{i}t(y_1-y_2)\sqrt{2}\}}{1+\phi(t)}\, \mathrm{d}t \right|^2 \end{aligned} \tag{A11b}$$

where ϕ is the error function $\phi(t) = (2/\sqrt{\pi}) \int_0^t \exp(-u^2)\, \mathrm{d}u$, one obtains

$$-\frac{1}{q^2} \int_0^\infty \mathrm{d}x\, x \int_{-\infty}^\infty \mathrm{d}y\, S(0, x; y) = \frac{1}{\sqrt{2}\, \pi^{3/2}} \tag{A12}$$

which differs from (A10) by factor 2. This discrepancy is intuitively associated with the slow power-law decay of correlations along the plane wall: an arbitrarily small deformation of the boundary towards the circle has a strong impact on this property. The analytical (Debye–Hückel approximation) and numerical (Monte Carlo simulation) studies of the surface charge correlations for finite Coulomb systems were given in ref. 27. The asymptotic form of these correlations is expected to depend on the shape of the plasma, but to be otherwise universal. The exact $\Gamma = 2$ solution for the "soft-wall" version of the 2d OCP with a quadrupolar field,[28] corresponding to a very large elliptic background, supports this finding.

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